Topics for Review for Midterm I in Calculus 10A

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1. Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is/are:

(1) a *function*? an *independent* variable; a *dependent* variable?
(2) a *domain*; a *range* of a function?
(3) *set notation* for a variety of intervals, for intersections, unions, and complements of sets, for subsets, for the empty set, for the set of even numbers or for odd numbers?
(4) the *graph* of a function? the *vertical line* test?
(5) a *piecewise defined* function?
(6) the function *absolute value* $|x|$?
(7) an *increasing* function; a *decreasing* function?
(8) an *odd* function; an *even* function?
(9) a *linear*, *quadratic*, *polynomial*, *rational* function? domains of these functions?
(10) a *power*, *exponential*, *logarithmic* function? domains of these functions?
(11) *one-to-one* (or injective) function? *onto* (or surjective) function?
(12) the *horizontal line* test?
(13) 1-1 *correspondence* (or bijective function)?
(14) *inverse* of a function? its graph in relation to the graph of the original function?
(15) a *trigonometric* function: sin$x$, cos$x$, tan$x$, cot$x$, and their inverses? domains of these functions?
(16) the *composition* $f \circ g$ of two functions $f(x)$ and $g(x)$?
(17) a *period* of a function? *doubling* or *halving* time for an exponential function?
(18) vertical and horizontal *shifts* of graphs of functions and when do they occur?
(19) important and frequently used *bases* for exponential functions?
(20) log-*plot* of a function? log-log-*plot*? why do we use them? what functions are good for such a plot?
(21) *limit* of $\lim_{x\to \Box_1} f(x) = \Box_2$ where each of the "boxes" can be a finite number or $\pm\infty$?

Example. To say that $\lim_{x\to -7} f(x) = 10$ means that:

(a) $f(x)$ can be made as close to 10 as we please, provided x is close enough to -7.
(b) every ϵ-goal around 10 can be achieved by $f(x)$ provided x is close enough to -7.
(c) $\forall \epsilon > 0$, $f(x)$ falls within ϵ of 10 provided x is close enough to -7.
(d) $\forall \epsilon > 0\ \exists \delta > 0$ such that $|f(x) - 10| < \epsilon$ whenever $-7-\delta < x < -7+\delta$, $x \neq -7$.

(22) *one-sided limit*, e.g. $\lim_{x\to a^-} f(x) = L$, $\lim_{x\to a^+} f(x) = -\infty$, etc.?
(23) *vertical asymptote* of $f(x)$ at a? *Horizontal asymptote* of $f(x)$? (Using limits, perhaps?)
(24) *What does it mean that a function $f(x)$:*

(a) is *continuous at* a? (Using limits, perhaps?)
(b) is *continuous on* (a, b)?
(c) has a *removable, jump* or *infinite discontinuity* at a?

(25) list of *elementary continuous* functions?
(26) the *tangent line* to the graph of a function $f(x)$ at $x = a$? a *secant line* of the graph of $f(x)$? what does it mean that $f(x)$ has a *vertical tangent* at $x = a$?
(27) the *average velocity* and the *instantaneous velocity* of an object whose movement is given by $f(t)$?
(28) the *derivative* $f'(a)$ at $x = a$? What does it mean that $f(x)$ is *differentiable* at a?
(29) the *derivative function* $f'(x)$? What does it mean that $f(x)$ is *differentiable* on (a, b)?

(30) *implicit differentiation*? When is it used?
(31) *critical* point of a function? *local* or *global extremum*? *inflection* point?
(32) *logarithmic derivative*? Where have we seen it and what is it good for?
(33) *related rates*? How do we find them?
(34) *linear* and *quadratic approximations* of a function? the *Taylor polynomial* of $f(x)$ at $x = a$?
(35) *Newton's method*? How does it compare to using Taylor polynomials?
(36) *indeterminacies*: quotient $0/0$, $\frac{\pm\infty}{\pm\infty}$, product $0 \cdot (\pm\infty)$, exponential $0^0, \infty^0, 1^\infty$, difference $(\infty - \infty)$?

2. Theorems and Methods

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems.

(1) **Finite Limits Laws (LLs)**: addition, subtraction, multiplication, division, basic examples, multiplication by a constant, powers, roots. (Be careful about the division law! What extra conditions does it require?) $x \to \square$ means: $x \to a$, $x \to \infty$, or $x \to -\infty$.

#	Theorem	Hypothesis	Conclusion
1	LL+	$\lim\limits_{x\to\square} f_1(x) = L_1,\ \lim\limits_{x\to\square} f_2(x) = L_2$	$\lim\limits_{x\to\square}(f_1(x)+f_2(x))=L_1+L_2$
2	LL$*c$	$\lim\limits_{x\to\square} f(x) = L,\ c \in \mathbb{R}$	$\lim\limits_{x\to\square}(cf(x))=cL$
3	LL$*$	$\lim\limits_{x\to\square} f_1(x) = L_1,\ \lim\limits_{x\to\square} f_2(x) = L_2$	$\lim\limits_{x\to\square} f_1(x)f_2(x) = L_1L_2$
4	LL$\div$	$\lim\limits_{x\to\square} f_1(x) = L_1,\ \lim\limits_{x\to\square} f_2(x) = L_2,\ f_2(x) \neq 0$ for $x \approx a,\ L_2 \neq 0$	$\lim\limits_{x\to\square} \dfrac{f_1(x)}{f_2(x)} = \dfrac{L_1}{L_2}$
5	LL$\circ$	$\lim\limits_{x\to a} f_1(x) = L_1,\ \lim\limits_{x\to L_1} f_2(x) = L_2,$	$\lim\limits_{x\to a} f_1(f_2(x)) = L_2$

(2) **Infinite Limit Laws (∞–LLs).**

Name	∞-LL: Formula	Example
1. addition	$\infty + \infty = \infty$	$\lim\limits_{x\to\infty}(x+x^2) = \lim\limits_{x\to\infty} x + \lim\limits_{x\to\infty} x^2 = \infty + \infty \overset{\infty\text{LL}}{=} \infty$
	$(-\infty)+(-\infty) = -\infty$	$\lim\limits_{x\to\infty}(-x-x^2) = \lim\limits_{x\to\infty}(-x) + \lim\limits_{x\to\infty}(-x^2) = (-\infty)+(-\infty) \overset{\infty\text{LL}}{=} -\infty$
	$\infty - \infty$ undefined	**Never use ∞-LLs in such cases.**
2. multiplication	$\infty \cdot \infty = \infty$	$\lim\limits_{x\to\infty} x^2 = \lim\limits_{x\to\infty} x \cdot \lim\limits_{x\to\infty} x = \infty \cdot \infty \overset{\infty\text{LL}}{=} \infty$
	$(-\infty)\cdot(-\infty) = \infty$	$\lim\limits_{x\to\infty}(-x)^2 = \lim\limits_{x\to\infty}(-x) \cdot \lim\limits_{x\to\infty}(-x) = (-\infty)\cdot(-\infty) \overset{\infty\text{LL}}{=} \infty$
	$\infty \cdot (-\infty) = -\infty$	$\lim\limits_{x\to\infty} -x^2 = \lim\limits_{x\to\infty} x \cdot \lim\limits_{x\to\infty}(-x) = \infty \cdot (-\infty) \overset{\infty\text{LL}}{=} -\infty$
3. multiplication by constant	$c \cdot \infty = +\infty$ if $c > 0$	$\lim\limits_{x\to\infty}(2x) = 2\lim\limits_{x\to\infty} x = 2 \cdot \infty \overset{\infty\text{LL}}{=} \infty$
	$c \cdot \infty = -\infty$ if $c < 0$	$\lim\limits_{x\to\infty}(-2x) = -2\lim\limits_{x\to\infty} x = -2 \cdot \infty \overset{\infty\text{LL}}{=} -\infty$
	$0 \cdot \infty$ undefined	**Never use ∞-LLs in such cases.**
4. addition by constant	$c + \infty = +\infty\ \forall c$	$\lim\limits_{x\to\infty}(-2+x) = -2 + \lim\limits_{x\to\infty} x = -2 + \infty \overset{\infty\text{LL}}{=} \infty$
	$c - \infty = -\infty\ \forall c$	$\lim\limits_{x\to\infty}(-2-x) = -2 - \lim\limits_{x\to\infty} x = -2 - \infty \overset{\infty\text{LL}}{=} -\infty$
5. division by $\pm\infty$	$\dfrac{c}{+\infty} = 0\ \forall c$	$\lim\limits_{x\to\infty} \dfrac{2}{x} = \dfrac{2}{\lim\limits_{x\to\infty} x} = \dfrac{2}{\infty} \overset{\infty\text{LL}}{=} 0$
	$\dfrac{c}{-\infty} = 0\ \forall c$	$\lim\limits_{x\to\infty} \dfrac{2}{-x} = \dfrac{2}{\lim\limits_{x\to\infty}(-x)} = \dfrac{2}{\infty} \overset{\infty\text{LL}}{=} 0$
6. division by $0^\pm$	$\dfrac{c}{0^+} = +\infty\ \forall c > 0$	$\lim\limits_{x\to 0} \dfrac{2}{x^2} = \dfrac{2}{0^+} \overset{\infty\text{LL}}{=} +\infty,\ \lim\limits_{x\to 0} \dfrac{-2}{x^2} = \dfrac{-2}{0^+} \overset{\infty\text{LL}}{=} -\infty$
	$\dfrac{c}{0^-} = -\infty\ \forall c > 0$	$\lim\limits_{x\to 0} \dfrac{2}{-x^2} = \dfrac{2}{0^-} \overset{\infty\text{LL}}{=} -\infty,\ \lim\limits_{x\to 0} \dfrac{-2}{-x^2} = \dfrac{-2}{0^-} \overset{\infty\text{LL}}{=} +\infty$
	$\dfrac{0}{0}, \dfrac{c}{0}$ undefined $\forall c$	**Never use ∞-LLs in such cases.**
7. basic	$\lim\limits_{x\to\infty} \dfrac{1}{x} = 0,\ \lim\limits_{x\to-\infty} \dfrac{1}{x} = 0,\ \lim\limits_{x\to\infty} x = \infty,\ \lim\limits_{x\to-\infty} x = -\infty$	
	$\lim\limits_{x\to 0^+} \dfrac{1}{x} = +\infty,\ \lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty,\ \lim\limits_{x\to 0} \dfrac{1}{x}$ **does not exist.**	

In the infinite limit laws, an expression like "$(-\infty) + (-\infty) = -\infty$" does **not** have a meaning on its own, except *in context*, i.e. it refers only to the following situation and to nothing else:

Theorem "$(-\infty) + (-\infty) = -\infty$". "If for functions $f(x)$ and $g(x)$ we know that $\lim_{x\to\Box} f(x) = -\infty$, $\lim_{x\to\Box} g(x) = -\infty$, then $f(x)+g(x)$ also has a limit when $x \to \Box$: this limit is $\lim_{x\to\Box}(f(x) + g(x)) = -\infty$."

Note that there are no infinite limit laws of the types $\infty - \infty, \infty/\infty, 0/0, 0 \cdot \infty$ since these symbolic expressions do not make sense, and they are called *indeterminacies*.

(3) **Continuity Laws (CLs).** Hypothesis for all continuity theorems below: If $f(x)$ and $g(x)$ are *continuous* at $x = a$, i.e. $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$, then

#	Theorem Name	Conclusion	Follows from
1	CL+	$f(x) + g(x)$ is also continuous at $x = a$	LL for sum
2	CL−	$f(x) - g(x)$ is also continuous at $x = a$	LL for difference
3	CL∗	$f(x)g(x)$ is also continuous at $x = a$	LL for product
4	CL÷ ($g(a) \neq 0$)	$f(x)/g(x)$ is also continuous at $x = a$	LL for ratio
5	CL∗c	$c \cdot f(x)$ is also continuous at $x = a$	LL for jumping constants
6	CL∘ ($h(x)$ is continuous at $b = f(a)$)	$h(f(x))$ is also continuous at $x = a$	LL for composition

Note that all *Continuity Laws (CLs)* follow from the corresponding *Limit Laws (LLs)*. The CLs above allow us to perform algebraic operations (and compositions) on continuous functions. Thus, we can construct more complex continuous functions from simpler continuous functions. To do this, we need to have a starting collection of

(4) **Basic Continuous Functions.** All functions below are *continuous* on their domains:

#	Function	Algebraic Formula and Conclusion	Follows from
1	Constants	c continuous at $\forall x$	LL for constants
2	Linear	$ax + b$ continuous at $\forall x$	LL for linear fn's
3	Quadratic	$ax^2 + bx + c$ continuous at $\forall x$	LL for quadratic fn's
4	Power	x^n continuous at $\forall x$, $\forall n = 1, 2, 3, ...$	LL for powers
5	Polynomial	$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ continuous at $\forall x$	LL for polynomials
6	Rational	$\dfrac{f(x)}{g(x)}$ continuous where $g(x) \neq 0$ ($f(x), g(x)$ - poly's)	LL for ratio, CL for poly's
7	Root	$\sqrt[n]{x}$ continuous at $\forall x$ where defined	LL for roots
8	Exponential	a^x continuous at $\forall x$ ($a > 0$)	LL for exponentials
9	Logarithmic	$\log_a x,\ \ln x$ continuous at $\forall x > 0$ ($a > 0$)	LL for logarithmics
10	Trigono-metric	$\sin x,\ \cos x$ continuous at $\forall x$; $\tan x,\ \cot x$ cont. on domain: $\tan x$: $x \neq \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots, (2n+1)\pi/2$, $\cot x$: $x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots, n\pi,\ n \in \mathbb{Z}$	LL for trig. fn's
11	Inverse Trigonomet-ric	arcsin$x$, arccos$x$, arctan$x$, arccot$x$ continuous on domain: arcsin$x$: $[-1, 1] \to [-\pi/2, \pi/2]$, arccos$x$: $[-1, 1] \to [0, \pi]$, arctan$x$: $\mathbb{R} \to (-\pi/2, \pi/2)$, arccot$x$: $\mathbb{R} \to (0, \pi)$	LL for inverse trig. fn's

(5) **Theorem (Diff. ⇒ cont.)** If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

(6) **Contrapositive Theorem. (Non-differentiable ⇒ non-continuous.)** If $f(x)$ is *not* continuous at a, then $f(x)$ is *not* differentiable at a.

(7) **Converse Statement is False!** Continuity does not guarantee differentiability. Counterexample?

(8) **Differentiation Laws (DLs)**

(a) **DL*c*:** $(c)' = 0$ for any constant c.

(b) **DL^c (Power Rule):** $(x^c)' = c\, x^{c-1}$ for any constant c.

(c) **DL∗*c*:** If $f(x)$ is a differentiable function, then $(c\, f(x))' = c\, f'(x)$.

(d) **DL$\pm$**: If $f(x)$ and $g(x)$ are differentiable, then their sum and difference are also differentiable: $(f(x)+g(x))' = f'(x)+g'(x)$, and $(f(x)-g(x))' = f'(x)-g'(x)$.

(e) **DL$*$ (PR)**: If $f(x)$ and $g(x)$ are differentiable, then their product is also differentiable: $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$.

(f) **DL$\div$ (QR)**: If $f(x)$ and $g(x)$ are differentiable at $x=a$ and $g(a)\neq 0$, then their quotient is also differentiable: $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$.

(g) **DL a^x**: $(e^x)' = e^x$ and $(a^x)' = a^x \cdot \ln a$ for any constant $a>0$.

(h) **DL $\log_a x$**: $(\ln x)' = \frac{1}{x}$ and $(\log_b x)' = \frac{1}{x\ln b}$ for all $x>0$ and any constant $b>0$.

(i) **DL trig**: $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \frac{1}{\cos^2 x}$, $(\cot x)' = \frac{-1}{\sin^2 x}$ (domains?)

(j) **DL arc-trig**: $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}$, $(\arctan x)' = \frac{1}{1+x^2}$, $(\text{arccot} x)' = \frac{-1}{1+x^2}$ (domains?)

(k) **DL^{-1}**: The derivatives of inverse functions $y=f(x)$ and $x=g(y)$ are reciprocal: $f'(x) = \dfrac{1}{g'(y)}$.

(l) **DL$\circ$ (CR)**: $(f(g(x))' = f'(g(x))\cdot g'(x)$.

(m) **CR shortcuts**: $(f^a(x))' = af^{a-1}(x)f'(x)$, $\dfrac{1}{f(x)} = -\dfrac{f'(x)}{f^2(x)}$, and $(\ln f(x))' = \dfrac{f'(x)}{f(x)}$.

(9) **L'Hospital's Rule (LH):** If $\lim_{x\to\square} f(x) = \lim_{x\to\square} g(x) = 0$ or $\lim_{x\to\square} f(x) = \pm\infty = \lim_{x\to\square} g(x)$, then we can attempt to find the LHS limit below by evaluating instead the RHS limit below:

$$\lim_{x\to\square}\frac{f(x)}{g(x)} \overset{\left(\frac{0}{0}\text{ or }\frac{\pm\infty}{\pm\infty}\right)}{=} \lim_{x\to\square}\frac{f'(x)}{g'(x)}.$$

If the limit on the RHS does not exist (or is more complicated to find), then try something else. If you get a determinate (i.e., if LLs work!), do NOT apply LH. If you get an indeterminate that is NOT a quotient one, then you need to work to rewrite the expression $\frac{f(x)}{g(x)}$ into (eventually) a quotient indeterminate in order to be able to apply LH.

(10) **Linear Approximation at** $x=a$: $f(x)\approx f(a)+f'(a)(x-a)$.

(11) **Quadratic Approximation at** $x=a$**:** $f(x)\approx f(a)+f'(a)(x-a)+\dfrac{f''(a)}{2}(x-a)^2$.

(12) **Taylor Polynomial of** $f(x)$ **at** $x=a$**:**

$$T_n(x) = f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

(13) **Famous Taylor Polynomials at** $a=0$:

$$\begin{aligned}
e^x &\approx 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{x^k}{k!} = T_k(x)\\
\sin x &\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^k\frac{x^{2k+1}}{(2k+1)!} = T_{2k+1}(x)\\
\cos x &\approx 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^k\frac{x^{2k}}{(2k)!} = T_{2k}(x)\\
\ln(1+x) &\approx \frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\cdots+(-1)^{k+1}\frac{x^k}{k} = T_k(x)
\end{aligned}$$

(14) **Famous Constants**[1] Obtained through Taylor Polynomials by Plugging in $x = 1$:

$$\begin{aligned}
e &\approx 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{k!} = T_k(1) \\
\sin 1 &\approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + (-1)^k \frac{1}{(2k+1)!} = T_{2k+1}(1) \\
\cos 1 &\approx 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots + (-1)^k \frac{1}{(2k)!} = T_{2k}(1) \\
\ln 2 &\approx \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots + (-1)^{k+1}\frac{1}{k} = T_k(1)
\end{aligned}$$

(15) **Newton's Method:** To approximate a root of a differentiable function, start close to the root with a guess of x_1, and consecutively apply the formula below to get closer and closer to the root by $x_1, x_2, x_3, \ldots, x_k$, etc.:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

3. Problem Solving Techniques

(1) **How do we find $\lim_{x\to a} f(x)$ when the (finite) LLs fail?**

(a) If $f(x) = \dfrac{g(x)}{h(x)}$ and "plugging in a" yields $\dfrac{0}{0}$, try factoring polynomials and rationalizing expressions with square roots. The idea is to end up with $(x-a)$ both in numerator and denominator, cancel it, and then again attempt to apply LLs.

(b) If $f(x)$ is a piecewise-defined function (i.e. given by different formulas on several intervals), try first to find the left-hand and the right-hand limits separately, and then compare them to see if they are equal (or if they exist, for that matter).

(c) If $f(x)$ is given by a formula involving absolute values, again proceed by finding and comparing the two one-sided limits.

(2) **How do we determine if a function is continuous at a?** By definition of continuity, there are 3 things to check:

(a) Find $\lim_{x\to a} f(x)$ by following either limits laws or the techniques suggested above. (If it doesn't exist, then the function has no chance of being continuous at a. If it exists but is an infinite limit $\pm\infty$, again the function is not continuous at a; in fact, it has an *infinite discontinuity* at a.)

(b) Find $f(a)$. If f is not defined at a, then the function is not continuous at a.

(c) If the above two steps yield two finite numbers, compare them to check if they are equal: $\lim_{x\to a} f(x) = f(a)$. If yes, the function is continuous at a; if not, the function is not continuous at a.

(3) **How do we determine if a function is continuous at a without using the definition of continuity?** We use CLs if applicable. For example, the function $f(x) = \frac{1}{x-3} \cdot \cos x + 6x^3$ is continuous as 2 because all comprising functions (rational, trigonometric and polynomial) are all continuous at 2. However, the function is not continuous at 3. (Why?)

(4) **How do we find $\lim_{x\to \square_1} f(x)$ when infinite limits are involved, but ∞–LLs fail?**

[1] For the super die-hards: can you obtain π through some Taylor polynomial? The answer, of course, is yes, but which function will be the right one?

(a) When finding the limit of a rational function: $\lim_{x\to\pm\infty}\dfrac{P_1(x)}{P_2(x)}$ (here $P_1(x)$ and $P_2(x)$ are polynomials), we know that ∞/∞ doesn't make sense. So, we factor out the highest powers of x from both top and bottom polynomials, cancel, and then apply LLs again. (Note: In the end, all that will matter will be the leading terms of the two polynomials - no other terms will survive the above operations.) Similar ideas apply to any other fractions which involve polynomials and possibly radicals.

(b) If ∞–LLs produce expressions involving $\infty - \infty$, we know that this doesn't make sense, so we look for a different approach. If polynomials are involved, factoring out the highest power is a good start. If square roots are involved, rationalizing might help. If two or more fractions are involved, putting them under a common denominator to arrive at one single fraction is the first step; then apply other techniques mentioned above.

(c) If ∞–LLs produce an expression of the type $0 \cdot \infty$, we know that this doesn't make sense, so we look for a different approach. Each example of this type has to be considered individually; most likely, we will end up factoring or rationalizing in search of common things to cancel, and after that we will attempt again to apply LLs.

(5) **How do we sketch graphs of functions $f(x)$?**

(a) If you recognize that the graph of $f(x)$ can be obtained from a graph of a well-known function via horizontal and/or vertical shifts, go for it! If the original (well-known) function already has vertical or horizontal asympotets, make sure to shift them too and indicate that the resulting function has these-and-these asymptotes.

(b) If the graph of $f(x)$ can't be obtained via the above shifts (e.g. $f(x)$ is a complicated function, or you just forgot how to sketch the graph of the "well-known" function), proceed as follows:

(i) First, look for "zeros" of the functions, i.e. for its x-intercepts: try to solve $f(x) = 0$ if possible. If $f(x)$ is a fraction, such solutions will be produced in the numerator. (The denominator will be irrelevant in this step.) It is also good to find the y-intercept, by setting $x = 0$ in $f(x)$.

(ii) Second, look for vertical asymptotes: these will appear where $f(x)$ has an infinite (at least) one-sided limit, i.e. if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$; then the vertical line $x = a$ is such an asymptote. If $f(x)$ is a fraction, such solutions will be produced by the roots of the denominator. (The numerator will be irrelevant in this step.)

(iii) Third, look for horizontal asymptotes: these will appear where $f(x)$ has a finite limit when $x \to \pm\infty$, i.e. if $\lim_{x\to\pm\infty} f(x) = L$; then the horizontal line $y = L$ is such an asymptote. If $f(x)$ is a fraction, both numerator and denominator will be involved in this step.

(iv) Finally, draw the vertical and horizontal asymptotes, mark the x-intercepts (and the y-intercept if applicable); draw the function so that it passes through the x- and y-intercepts and respects all asymptotes as found above. Be careful nearby the vertical asymptotes to reflect whether a given one-sided limit is $+\infty$ or $-\infty$, correspondingly. It doesn't hurt to plot several other points in each interval between asymptotes and x-intercepts, to make your graph more precise and make sure you haven't done any silly calculation mistakes in the above steps.

(6) **How do we prove statements about limits using the limit definitions?**

We pray that such a problems is not on the midterm. If this doesn't help, we follow the steps below.

(a) Consider the type of limit you are given: $\lim_{x\to\square_1} f(x) = \square_2$ and decide what your goals will be depending on what $\square_2$ is.

- If $\square_2 = L$ - a finite number, then your function "wants to be close to this number L"... how close? - ϵ-close. Your goals will be therefore ϵ-goals around L, which can be written in one of the following three equivalent forms:

$$f(x) \text{ is within } \epsilon \text{ of } L \;\Leftrightarrow\; L-\epsilon < f(x) < L+\epsilon \;\Leftrightarrow\; |f(x)-L| < \epsilon$$

- If $\square_2 = +\infty$, then your function "wants to be close to $+\infty$"... To say "ϵ-close to $+\infty$" doesn't make a whole lot of sense! Instead, we want $f(x)$ to get as large as we please. Hence, our goals will be M-goals, where $M > 0$. Such a goal can be written simply as $f(x) > M$.
- If $\square_2 = -\infty$, then your function "wants to be close to $-\infty$"... To say "ϵ-close to $-\infty$" doesn't make a whole lot of sense! Instead, we want $f(x)$ to get as small as we please. Hence, our goals will be M-goals, where $M < 0$. Such a goal can be written simply as $f(x) < M$.

(b) Decide next what type of answers you are looking for depending on what $\square_1$ is.

- If $\square_1 = a$ - a finite number, then x "wants to be close to this number a"... how close? - δ-close. Your answers will be therefore δ-intervals around a, which can be written in one of the following three equivalent forms:

$$x \text{ is within } \delta \text{ of } a \;\Leftrightarrow\; a-\delta < x < a+\delta \;\Leftrightarrow\; |x-a| < \delta.$$

- If $\square_1 = +\infty$, then x "wants to be close to $+\infty$"... To say "δ-close to $+\infty$" doesn't make a whole lot of sense! Instead, we want x to be large enough. Hence, our answers will be M-answers, where $M > 0$, written simply as $x > M$.
- If $\square_1 = -\infty$, then x "wants to be close to $-\infty$"... To say "δ-close to $-\infty$" doesn't make a whole lot of sense! Instead, we want x to be small enough. Hence, our answers will be M-answers, where $M < 0$, written simply as $x < M$.
- **Summary of (ϵ, δ)–Definition Types of Goals and Answers.** Here are all 9 possible types of limits (3 possible goals, and 3 possible answers) for $\lim_{x\to\square_1} f(x) = \square_2$:

limit $\square_2$	goal for $f(x)$	$x \to \square_1$	answer for x
L	ϵ-goal around L	$x \to a$	δ-interval around a
$+\infty$	M-goal, $M > 0$	$x \to +\infty$	N-answer, $N > 0$
$-\infty$	M-goal, $M < 0$	$x \to -\infty$	N-answer, $N < 0$

(c) Put together your goals and your types of answers to see what you are really after. And then proceed either algebraically or graphically.

(i) The algebraic way is more rigorous, but then less insightful. It goes as follows.

Step I: Conjecture your limit.

Step II: Write your goal for your function, e.g. we want $5-\epsilon < f(x) < 5+\epsilon$ (here $\square_2 = L = 5$), or $f(x) < M$ (here $\square_2 = -\infty$ and $M < 0$). Next, try to solve these inequalities for x. Keep in mind that sometimes considerations like $x > 0$ or $x < 0$ can help eliminate irrelevant information. Next, put your answer in the form of δ-interval around a, or in the form of $x > M$ or $x < M$, depending on what type of answer you are looking for. (Recall that sometimes when looking for δ, we have to take the smaller of two distances from a in order to ensure that our δ-interval is centered at a.) Make sure you announce what your final δ or M answer is, do NOT forget to state your conclusion: The above steps show that indeed blah-blah ... ($\lim_{x\to\square_1} f(x) = \square_2$)

(ii) The geometric way is less rigorous, but then more intuitive. Start by sketching a graph of your function (you may have to shift graphs of simpler functions or plot several points for your function). Then mark your goal: either an ϵ-strip around L, or an M-region (above or below M depending on whether your limit is $+\infty$ or $-\infty$.)

Mark the portion of the graph which lies inside your goal area. Project this portion onto the x-axis, i.e. mark all x's above which $f(x)$ falls into the goal area. These x's should be grouped in one or more intervals. Make sure you eliminate the irrelevant intervals for x (e.g. if $x \to -\infty$, then intervals with positive x's are irrelevant; if $x \to 7$, then an interval around 2 is probably also irrelevant, but an interval around 7 will be VERY relevant!) You should be left with only one relevant interval I for x.

Using the formula for the function $f(x)$, find precisely what this I is. This usually entails the following calculations: $f(x) = L + \epsilon$ (when the function enters/exits "from above" the ϵ-strip around L); or $f(x) = L - \epsilon$ (when the function enters/exits "from below" the ϵ-strip around L); or $f(x) = M$ (when the function enters/exits the M-goal region). Solve this for x to find your "good interval" I - check with your graph to make sure that what you obtain makes sense.

Finally, translate this into the type of answer you are expected to obtain: $a - \delta < x < a + \delta$, or $x < M$ or $x > M$, and state this answer clearly. (Make sure that all of the above calculations and work with the graph is recorded properly in your solution!) Conclude by stating that therefore blah-blah... ($\lim_{x \to \Box_1} f(x) = \Box_2$)

(7) **How do we find derivatives from the definition?** Read carefully if you are being asked to find a specific derivative $f'(a)$, or the *whole* derivative function $f'(x)$. In each case, you have two choices how to proceed, as listed below.

(a) $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} \cdot$ Here a is a constant and x moves towards a, so we expect that x will disappear and a will remain in the final result for $f'(a)$.

(b) $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} \cdot$ Here a is a constant and h moves towards 0, so we expect that h will disappear and a will remain in the final result for $f'(a)$. This formula is nothing else but formula (a) where x is replaced by $a + h$.

(c) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} \cdot$ Here x is viewed as a constant and h moves towards 0, so we expect that h will disappear and x will remain in the final result for the derivative function $f'(x)$. This formula is nothing else but formula (b) where a is replaced by x.

(d) One can also find $f'(x)$ by first finding $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$; in the result of this calculation x will disappear and only a will remain; in this final formula replace a by x to obtain a formula for the whole derivative function $f'(x)$.

(8) **How do we find equations of tangent lines?**

(a) First find the corresponding derivative $f'(a)$: this will be the slope of your tangent line.

(b) Next use the point–slope formula for the point $P(a, f(a))$ and for the slope $f'(a)$ from part (a):

$$f'(a) = \frac{y - f(a)}{x - a} \quad \Leftrightarrow \quad y - f(a) = f'(a)(x - a) \cdot \tag{1}$$

Here a and $f(a)$ are constants, and x and y are variables in the equation for your tangent line. Where suitable, multiply through and simplify to obtain a formula of the type:

$$y = f'(a) \cdot x + b.$$

(9) **How do we find all the tangents lines to the graph of $f(x)$ which are parallel to some line $y = mx + b$?**

(a) First find the slope m of the given line; be careful with this since the line may not be given in the standard "line-equation" form as above and you may have to rewrite it.

(b) Find the derivative $f'(x)$.

(c) Next, set $f'(x) = m$, where m is the found slope of the line above, and solve it for x.

(d) Finally, let's say you found several solutions x_1, x_2... etc. What remains is to find the corresponding points on the graph of $f(x)$ through which the wanted tangent lines will pass: $P_1(x_1, f(x_1))$, $P_2(x_2, f(x_2))$, etc. If the problem asks for finding the equation of these tangent lines, well, proceed now with the point-slope formula as before.

(10) **For the hot-shots only:**
How do we find all tangent lines to the graph of $f(x)$ passing through a point (c, d)?

(a) First find the derivative $f'(x)$.

(b) Next, set the point-slope formula for the tangent line:

$$f'(a) = \frac{y - f(a)}{x - a}.$$

You will not know at this moment what a is (that's what you want to find in the end), nor what x or y are (these are the variables in the equation of the tangent line).

(c) Substitute the point (c, d) into the above equation ((c, d) is supposed to lie on your tangent line, hence substituting $x = c$ and $y = d$ must work):

$$f'(a) = \frac{d - f(a)}{c - a}.$$

At this point, you must realize that only a is left "unknown" in this equation; everything else must be a number; thus, we can solve this equation for a.

(d) Now that you know what a is (are), find the equation of the corresponding tangent line(s) at $(a, f(a))$. The problem may be asking for less: just find the points of tangency $(a, f(a))$.

(11) **How do we sketch graphs of the derivative function $f'(x)$ given the graph of $f(x)$?**

(a) Find where the given function $f(x)$ is *not* differentiable; at these x's $f'(x)$ will not exist. There are many different reasons for $f'(x)$ not to exist. Here follow some such reasons:

- $f(x)$ is *not* defined at $x = a$. Then we can't even talk about the derivative at $x = a$.
- $f(x)$ is defined at $x = a$ but is *not* continuous there. Then the contrapositive theorem implies that $f(x)$ is *not* differentiable at $x = a$. No matter what type of discontinuity $f(x)$ has at $x = a$, $f'(a)$ will *not* exist. An infinite discontinuity of $f(x)$ (i.e. $f(x)$ has a vertical asymptote $x = a$) usually translates into a vertical asymptote for $f'(x)$ at $x = a$. A jump or removable discontinuity of $f(x)$ usually translates into a jump or removable discontinuity for $f'(x)$. Each case is treated separately to see what happens with $f'(x)$.
- $f(x)$ is defined and continuous at $x = a$, but is *not* "smooth" there, i.e. has a cusp(corner). Usually here either the two one-sided tangents exist at $x = a$ but have different slopes, or there is a vertical tangent at $x = a$. In the former case, this will translate into a jump discontinuity of $f'(x)$; in the latter case, this translates into a vertical asymptote of $f'(x)$.
- $f(x)$ looks "smooth" at $x = a$, but has a vertical tangent there. Again, this will translate into a vertical asymptote of $f'(x)$.

(b) After marking all x's where $f'(x)$ does not exist (including possible vertical asymptotes, etc.), we move on to graphing $f'(x)$ where it exists. First find all places where the tangents to $f(x)$ are horizontal and mark the corresponding 0's on the graph of f'. Next determine the intervals where $f(x)$ increases i.e. has positive tangent slopes, and where $f(x)$ decreases, i.e. has negative tangent slopes. In the former case, $f'(x)$ will be positive, and in latter case, $f'(x)$ will be negative. In each such interval, answer the following two questions: whether the tangent slopes are positive or negative, and whether the tangent slopes themselves are increasing or decreasing. Translate this into the corresponding property of $f'(x)$.

(c) For more precise drawing, in each of the above intervals mark several tangent lines, guestimate their slopes and mark the corresponding points on the graph of $f'(x)$. Connect all these marked points to obtain the graph of $f'(x)$. Don't forget the places where $f'(x)$ was not defined!

(12) **How do we find derivatives using DLs?** If you are given $f(x)$ via one formula and you are not asked to use the definition of derivative, you apply DLs. However, for the purposes of this Midterm 1, the only DLs allowed are listed previously in this handout: DLs for polynomials, power rule, DLs for addition, difference and multiplication by a constant.

(a) First see if you can further simplify the given formula. In particular, try to avoid applying the Quotient Rule whenever possible because it is prone to errors. In practice this mean: try to get rid of denominators by either splitting fractions and then simplifying each fraction separately (see formulas for fraction manipulations below), or by direct cancellation of common stuff in the numerator and denominator, or by moving the denominator into the numerator: e.g. x^3 in the denominator becomes x^{-3} in the numerator.

(b) If you are going to apply the Power Rule, turn all expressions like $\sqrt[n]{x}^m$ into the standard form $x^{\frac{m}{n}}$. Again, such expressions in the denominator should move into the numerator wherever suitable by flipping the sign of the power: $\sqrt[n]{x}^m$ in the denominator becomes $x^{-\frac{m}{n}}$ in the numerator.

(c) Look at your function $f(x)$ to figure out its components, the simpler pieces it is made of, and decide which DL(s) you are going to use. In some cases, you may have to apply several DLs one after the other, so keep good track of your intermediate results, or else your calculations will be untraceable. A good strategy is to name some of the simpler components of $f(x)$, e.g. $g(x)$, $h(x)$, etc. and perform some of the necessary differentiation on these functions on the side and then put back your results together. To reduce errors and to make clear that you do know the DLs, it is always good to write the DL formula in terms of functions at first, e.g.

$$((5x+2)\cdot x^3)' \overset{PR}{=} (5x+2)'\cdot x^3 + (5x+2)\cdot (x^3)' = ...$$

4. Useful Formulas and Miscellaneous Facts

(1) **Quadratic formula**: useful for factoring quadratic polynomials as $a(x-x_1)(x-x_2)$, where x_1 and x_2 are the two roots of the polynomial, and a is the leading coefficient. Useful also for graphing quadratic polynomials: will yield the x-intercepts (or tell you that they don't exist.)

(2) **Rationalizing formula**: $\sqrt{A}-\sqrt{B} = \dfrac{(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B})}{\sqrt{A}+\sqrt{B}} = \dfrac{(A-B)}{\sqrt{A}+\sqrt{B}}$.

(3) **Factorization formulas**: $A^2-B^2=(A-B)(A+B)$ and $A^3-B^3=(A-B)(A^2+AB+B^2)$.

(4) **Binomial formulas:** $(A+B)^2 = A^2+2AB+B^2$, $(A+B)^3 = A^3+3A^2B+3AB^2+B^3$.

(5) **Putting fractions under a common denominator**. The most general formula is as follows: $\frac{A}{B} + \frac{C}{D} = \frac{AD+BC}{BD}$. Yet, it is worth noting that if fractions already share something in their denominators, it will be faster to take this into account, e.g.

$$\frac{2x+1}{x^2} + \frac{x^3}{x(x-1)} = \frac{(2x+1)(x-1) + x \cdot x^3}{x^2(x-1)} = \frac{x^4+2x^2-x-1}{x^2(x-1)}.$$

(6) **Absolute value inequalities**. Note the following expressions which mean the same things:

(a) $|x| < A \;\Leftrightarrow\; -A < x < A$;

(b) $|x-B| < A \;\Leftrightarrow\; -A < x - B < A \;\Leftrightarrow\; B - A < x < B + A \;\Leftrightarrow\; x \in (B-A, B+A)$;

(c) $|3x^2 - 10| < 5 \;\Leftrightarrow\; -5 < 3x^2 - 10 < 5 \;\Leftrightarrow\; 5 < 3x^2 < 15 \;\Leftrightarrow\; \sqrt{5/3} < |x| < \sqrt{15/3}$

$\Leftrightarrow\; \sqrt{5/3} < x < \sqrt{15/3}$ (when $x \geq 0$), or $\sqrt{5/3} < -x < \sqrt{15/3}$ (when $x < 0$).

As a final answer, the original inequality is satisfied when $x \in (\sqrt{5/3}, \sqrt{5}) \cup (-\sqrt{5}, -\sqrt{5/3})$.

(d) $|2f(x) - 7| < 0.5 \;\Leftrightarrow\; -0.5 < 2f(x) - 7 < 0.5 \;\Leftrightarrow\; 3.25 < f(x) < 3.75 \;\Leftrightarrow\; f(x) \in (3.25, 3.75)$.

(7) **Trigonometric functions**: domains of definition, ranges, graphs, periods; where they increase, decrease, values at $x = 0, \pi/3, \pi/4$ and so on "prominent" numbers; vertical asymptotes (if any); trigonometric identities; radians versus degrees.

(8) **Exponential and Logarithmic functions**: domains of definition, ranges, graphs; for which bases do these function increase/decrease, c^x and $\log_c x$ are inverse functions of each other, basic identities.

(9) **Manipulations with Fractions**

(a) *Splitting fractions:* $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$; $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{d}$;

(b) **Wrong** *formula:* $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$.

(c) *Putting fractions under a common denominator:* $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

(d) *When denominators have something in common:* $\frac{a}{be} + \frac{c}{de} = \frac{ad+bc}{bde}$.

(e) *"Fractions over fractions":* $\frac{a}{c} : \frac{b}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$; $\frac{a}{\frac{c}{d}} = \frac{ad}{c}$; $\frac{\frac{a}{b}}{c} = \frac{a}{bc}$.

I cannot conceive of any other operation on fractions! If you think of one, let me know!

(10) **Manipulations with Exponentials and Logarithms**

(a) $a^{b+c} = a^b \cdot a^c$, $\frac{a^b}{a^c} = a^{b-c}$, $(a^b)^c = a^{bc}$, $a^{\frac{b}{c}} = \sqrt[c]{a^b}$, $\frac{1}{\sqrt[c]{a^b}} = \frac{1}{a^{\frac{b}{c}}} = a^{-\frac{b}{c}}$, $a^0 = 1$.

(b) $\ln(e^x) = x$, $e^{\ln x} = x$, $\ln(a^b) = b \ln a$.

(11) **Trigonometric Formulas**

(a) $\sin^2 x + \cos^2 x = 1$;

(b) $\tan x = \frac{\sin x}{\cos x}$; $\cot x = \frac{\cos x}{\sin x}$;

(c) $\sin(\frac{\pi}{2} - x) = \cos x$; $\cos(\frac{\pi}{2} - x) = \sin x$;

(d) the values of $\sin x$, $\cos x$, $\tan x$ and $\cot x$ at all "prominent" x's: $0, \pi/6, \pi/4, \pi/3, \pi/2, \pi$, etc.

5. Cheat Sheet

For the midterm, you are allowed to have a "cheat sheet" - *one page* (which means only on one side) of a regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in disqualifying your midterm. I may decide to randomly check your cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.